The Radiation of Electromagnetic Power by Microstrip Configurations

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***Abstract*—A new technique for calculating the radiation losses of microstrip configurations is presented. The method applies if the wavelength is large compared to the width of the conducting strip and the thickness of the dielectric wafer. It is shown that the radiated power, which is partly carried by "space waves" and partly by "surface waves," can be computed in terms of the specific inductance and the specific capacitance of the transmission line, without making any assumptions regarding the current distribution in the microstrip. It appears that the fraction of the radiated power carried by surface waves contains the frequency to a higher power than does the fraction carried by space waves and is therefore relatively small. The investigated configurations are the infinitely long transmission line excited by a voltage-slit, the half-wavelength straight resonator, the full-wavelength circular resonator, and the quarter-wavelength hairpin resonator. It follows that the quality factor of the straight resonator and the circular resonator are inversely proportional to the square of the frequency, whereas the quality factor of the hairpin resonator is inversely proportional to the fourth power of the frequency.**

Introduction

A RIGOROUS CALCULATION of the power radiated by microstrip configurations is very complicated. This is due to the fact that the problem of finding the current distribution in the conducting strip and in the short-circuiting posts is a mixed boundary value problem, which cannot be solved by analytical methods. However, in most cases of practical interest the wavelength of the propagating mode is large compared to the width of the conducting microstrip and to the thickness of the dielectric wafer. In such cases detailed knowledge of the current distributions is not required for calculating the radiated power. In fact it can be shown that, in the long-wavelength limit, only the total current flowing along the microstrip and in the short-circuiting posts is relevant for power radiation. These total currents can be calculated by a simple transmission line approach, using the concept of the specific capacitance and the specific inductance of the microstrip. We shall assume in this paper that the long-wavelength condition is satisfied and that the specific capacitance C and the specific inductance L are known parameters.

A rather classical situation is met if the dielectric and magnetic properties of the wafer are not different from those in the adjacent halfspace. In that case the far-field Hertzian vector can be obtained from the well-known solution of Poisson's equation, which is valid in an unbounded and uniform medium. However, the dielectric constant of the wafer is normally many times larger than that in the adjacent halfspace and in that case the "uniform medium" approach is not satisfactory. An approach followed by Lewin [1] and others [3]–[7] is to account for the larger dielectric constant of the wafer by introducing "polarization" currents, flowing from the strip to the ground plane.

The magnitude of these currents is estimated by Lewin from an approximate electric field configuration under the strip, together with an "effective" dielectric constant of the platelet. The Hertzian vector on a hemisphere of infinite radius is then computed by taking as the source function the electric currents and the polarization currents in the platelet and by using the Green's function of the unbounded uniform medium. The total radiated power is next obtained by integrating Poynting's vector over the hemisphere of infinite radius. Though we may expect that this approach will give the correct order of magnitude of the radiated power, the errors introduced by the simplifying assumptions seem hard to estimate.

In the approach followed in this paper, use has been made of the fact that the power radiated by the microstrip configuration should be equal to the power necessary to maintain the current density at a stationary value. This furnished power, in turn, can be found by calculating the scalar product of the current density and the complex conjugate of the electric field opposing the current density and integrating this scalar product over the space coordinates. The time average of the total furnished power is then obtained by taking the opposite of the real part and dividing by two. However, in space coordinates this calculation is very complicated, due to the complicated structure of Green's function interrelating the electric field and current density in an inhomogeneous medium. A much simpler expression for the delivered power is obtained in terms of the Fourier transforms of current density and electric field with respect to the coordinates of the plane of the wafer. The interrelation between these Fourier transforms is simply algebraic, because of the translational symmetry of the dielectric wafer. The so-called impedance dyadic giving this interrelation is found to be an elementary function of the wafer parameters. As a result of this Fourier transformation the calculation of the furnished power can be carried out without any simplification of the model, the only assumption being that the long-wavelength condition is satisfied. In the next section we give a detailed description of the method.

Outline of the Method

We denote the x component of the surface current density in the plane $z = 1$ by j_1 and similarly the y component by j_2 (see Fig. 1). The current density in the region $0 < z < 1$ is

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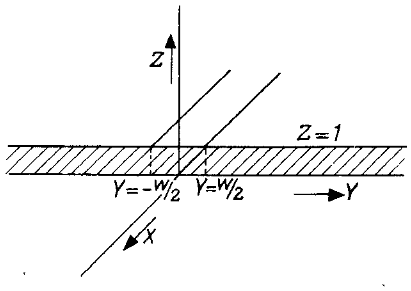

Fig. 1. Cross-sectional view of a microstrip transmission line.

assumed to be in the z direction and independent of z. This constraint has no influence on the power radiation, provided that the long-wavelength condition is satisfied. We denote this current density by j_3 and we represent the scalars j_1, j_2, and j_3 simultaneously by the vector $\boldsymbol{j}(x,y)$.

Similarly, let E_1 represent the x component of the electric field in the plane $z = 1$ and E_2 the y component. The average value of the z component of the electric field in the region $0 < z < 1$ is denoted by E_3. We represent E_1, E_2, and E_3 by the vector $\boldsymbol{E}(x,y)$. We denote the time average of the power to be furnished to maintain $\boldsymbol{j}$ stationary by W. By definition, W is given by

$$W = -\tfrac{1}{2} \operatorname{Re} \iint_{-\infty}^{\infty} \boldsymbol{E}^* \cdot \boldsymbol{j} \, dx \, dy. \tag{1}$$

In (1) the asterisk indicates the complex conjugate value. We note that the plane $z = 0$ does not contribute to W, because at $z = 0$ the x and y components of the electric field are zero. This is not necessarily the case in the conductive strip, in spite of the assumed infinite conductivity, because an externally applied driving field may be present. The infinite conductivity then assures that the sum of the electric and the driving field is zero [2]. In order to evaluate (1) we introduce the Fourier transforms $\boldsymbol{E}''$ and $\boldsymbol{j}''$ of $\boldsymbol{E}$ and $\boldsymbol{j}$:

$$\boldsymbol{E}(x,y) = \iint_{-\infty}^{\infty} \boldsymbol{E}''(\alpha,\beta) \exp(-i\alpha x - i\beta y) \, d\alpha \, d\beta$$

$$\boldsymbol{j}(x,y) = \iint_{-\infty}^{\infty} \boldsymbol{j}''(\alpha,\beta) \exp(-i\alpha x - i\beta y) \, d\alpha \, d\beta. \tag{2}$$

Application of Parseval's theorem to (1) then yields

$$W = -\tfrac{1}{2} \operatorname{Re} 4\pi^2 \iint_{-\infty}^{\infty} \boldsymbol{E}''^* \cdot \boldsymbol{j}'' \, d\alpha \, d\beta. \tag{3}$$

In order to eliminate the unknown $\boldsymbol{E}''$ from (3) we next establish the interrelation between $\boldsymbol{E}''$ and $\boldsymbol{j}''$. As a result of the translational symmetry of the dielectric wafer the three components of $\boldsymbol{E}''$ are related to those of $\boldsymbol{j}''$ by three linear algebraic equations. We write these equations in the following form:

$$E_k'' = \sum_{l=1}^{3} Z_{kl} j_l'', \qquad k = 1,2,3$$

or, in matrix notation

$$\boldsymbol{E}'' = Z \cdot \boldsymbol{j}''. \tag{4}$$

The elements of the "impedance dyadic" Z can be found by elementary methods. The computation of the elements Z_{11}, $Z_{12} = Z_{21}$, and Z_{22} is given in [2]. The elements Z_{31} and Z_{32} can analogously be found from the set of equations (A1)–(A13) in [2]. The dyadic elements Z_{13}, Z_{23}, and Z_{33}, on the other hand, can be obtained in the following way.

We replace the last equation (A2) with the inhomogeneous Maxwell's equation

$$\operatorname{curl}(\boldsymbol{H}) = \varepsilon \, \partial \boldsymbol{F}/\partial t + j_3 \boldsymbol{i}_3, \qquad 0 < z < 1$$

where $\boldsymbol{i}_3$ is the unit vector in the z direction. The boundary conditions at $z = 1$ are conveniently expressed by

$$(H_2)_{z=1-0} - (H_2)_{z=1+0} = j_1$$

$$(H_1)_{z=1+0} - (H_1)_{z=1-0} = j_2.$$

Substitution of the particular solution

$$\begin{aligned}
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= \exp(i\omega t - i\alpha x - i\beta y), \qquad 0 < z < 1 \\
H_1 &= (\beta/\omega\mu) F_3 \\
H_2 &= -(\alpha/\omega\mu) F_3 \\
H_3 &= 0 \\
\boldsymbol{F} &= \boldsymbol{H} = 0, \qquad z > 1
\end{aligned}$$

into (A1) and (A2) and application of the above boundary conditions yield three additional relations between the dyadic elements, from which Z_{13}, Z_{23}, and Z_{33} can be solved. We find

$$\begin{aligned}
Z_{11} &= -\frac{\alpha^2\gamma^2}{i\omega\varepsilon(\alpha^2+\beta^2)} F_1 - \frac{i\omega\beta^2\mu}{\alpha^2+\beta^2} F_2 \\
Z_{12} = Z_{21} &= -\alpha\beta \left\{ \frac{\gamma^2}{i\omega\varepsilon(\alpha^2+\beta^2)} F_1 - \frac{i\omega\mu}{\alpha^2+\beta^2} F_2 \right\} \\
Z_{13} = -Z_{31} &= -\frac{\alpha}{\omega\varepsilon} F_1 \\
Z_{22} &= -\frac{\beta^2\gamma^2}{i\omega\varepsilon(\alpha^2+\beta^2)} F_1 - \frac{i\alpha^2\omega\mu}{\alpha^2+\beta^2} F_2 \\
Z_{23} = -Z_{32} &= -\frac{\beta}{\omega\varepsilon} F_1 \\
Z_{33} &= -\frac{i\omega\mu}{\gamma^2} - \frac{\alpha^2+\beta^2}{\gamma^2 i\omega\varepsilon} F_1
\end{aligned} \tag{5}$$

with

$$F_1 = \left\{ \frac{\gamma}{\tanh\gamma} + \frac{\gamma^2}{\varepsilon\gamma_0} \right\}^{-1}$$

$$F_2 = \left\{ \frac{\gamma}{\tanh\gamma} + \mu\gamma_0 \right\}^{-1}$$

$$\gamma^2 = \alpha^2 + \beta^2 - \varepsilon\mu\omega^2$$

$$\gamma_0^2 = \alpha^2 + \beta^2 - \omega^2.$$

In (5), ε and μ represent the permittivity and the permeability, respectively, of the dielectric wafer relative to those in the half-space $z > 1$. The permittivity and permeability in the half-space $z > 1$ are normalized to unity. The variables $\boldsymbol{E}$ and $\boldsymbol{j}$ depend on time through the factor exp $(i\omega t)$. Substitution of (4) into (3) yields

$$W = \tfrac{1}{2}\,\mathrm{Re} \iint_{-\infty}^{\infty} I(\alpha,\beta)\, d\alpha\, d\beta \tag{6}$$

where

$$I(\alpha,\beta) = -4\pi^2 \boldsymbol{j}''^* \cdot Z^* \cdot \boldsymbol{j}''.$$

It is assumed that α and β are real variables. However, for the following two reasons there is an ambiguity in W, given by (6). First, as we shall presently investigate in more detail, for real values of ω, $I(\alpha,\beta)$ may become infinitely large in some regions of the α–β plane. In order to remove this ambiguity we note that if we admit a current distribution that, in the time domain, remains nonzero for $t \to \infty$, we must, in the frequency domain, impose on ω the condition

$$\mathrm{Im}\,(\omega) \leq -0 \tag{7}$$

otherwise a Fourier transformation with respect to time is not allowed. Second, ambiguity arises from the indeterminateness of the sign of γ_0. This difficulty is solved by the following argument. The matrix elements of Z, given by (5), are derived by assuming in the region $z > 1$ plane-wave solutions that depend on time and space through the factor exp $(i\omega t - i\alpha x - i\beta y - \gamma_0 z)$. However, in order to satisfy Sommerfeld's radiation condition we must impose on γ_0 the auxiliary condition

$$\mathrm{Re}\,(\gamma_0) \geq +0. \tag{8}$$

In view of (7) and (8) the ambiguity in (6) has now disappeared. We note that the uncertainty of the sign of γ introduces no ambiguity because Z is an even function of γ.

From the law of conservation of energy it follows that, for Im $(\omega) = -0$, W must be equal to the time average of the power passing across a hemisphere in the half-space $z > 0$ with a radius sufficiently large to cover the region of power supply. This power is partly carried by "space waves," i.e., plane waves propagating in the half-space $z > 1$; partly by "surface waves," guided by the dielectric wafer; and, if the microstrip is infinitely long, partly by the microstrip itself.

We shall now show that each of these contributions can be attributed to distinct regions in the α–β plane. To that end we first remark that (6) is valid for any arbitrary current distribution. Let us in particular consider the infinitesimal distribution $d\boldsymbol{j}''(\alpha,\beta)$, defined by

$$d\boldsymbol{j}''(\alpha,\beta) = \boldsymbol{j}''(\alpha,\beta),\ \alpha_0 < \alpha < \alpha_0 + d\alpha_0,\ \beta_0 < \beta < \beta_0 + d\beta_0$$

$$d\boldsymbol{j}''(\alpha,\beta) = \boldsymbol{0}, \qquad \text{elsewhere.}$$

Then, in view of (6), the furnished power dW is

$$dW = \tfrac{1}{2}\,\mathrm{Re}\,\{I(\alpha_0,\beta_0)\, d\alpha_0\, d\beta_0\}.$$

On the other hand the emitted power dW passing across the "large" hemisphere in the half-space $z > 0$ is carried exclusively by those plane waves whose x component k_x of the wave vector lies in the interval $\alpha_0 < k_x < \alpha_0 + d\alpha_0$ and, analogously, whose y component k_y lies in the interval $\beta_0 < k_y < \beta_0 + d\beta_0$. In other words, $I(k_x,k_y)$ can be interpreted as the density in the two-dimensional wavenumber space k_x,k_y of the emitted power W. Now, let us first consider the part of the power radiated into the half-space $z > 1$. This part is carried by plane waves whose z component of the wave vector is equal to $-i\gamma_0$. Such plane waves are nonevanescent only if γ_0 is purely imaginary. Hence, in view of (5), this part of the power should be attributed to the region in the α–β plane given by

$$\alpha^2 + \beta^2 \leq \omega^2.$$

We next consider the part of the radiated power carried by surface waves. It is typical of such surface waves that $\boldsymbol{j}''$ is zero, whereas $\boldsymbol{E}''$ is finite. Hence the power carried by surface waves should be attributed to that part of the α–β plane for which $1/\mathrm{Det}\,(Z) = 0$. An investigation of (5) reveals that at $\alpha = 0$, $\beta = 0$ Det (Z) remains finite. On the other hand Det (Z) becomes infinitely large if either F_1^{-1} or F_2^{-1} in (5) goes to zero.

For the case where

$$F_1^{-1} = 0 \tag{9}$$

we easily verify from (4) and (5) that $E_1''/E_2'' = \alpha/\beta$. Hence the electric vector in the plane $z = 1$ is parallel to the direction of propagation, i.e., the surface waves are of the TM type. We note that in the long-wavelength limit we may replace tanh $(\gamma)/\gamma$ by unity. Hence, in our case, equation (9) is equivalent to

$$\alpha^2 + \beta^2 = \omega^2 + \{(\varepsilon\mu - 1)/\varepsilon\}^2\omega^4. \tag{10}$$

From this we conclude that the power carried by surface waves of the TM type corresponds to a circle in the α–β plane, determined by (10). We note that (10) also determines the propagation velocity of the type of waves considered, i.e., equation (10) is the so-called dispersion relation.

For the case where

$$F_2^{-1} = 0 \tag{11}$$

we find, by an analogous reasoning, that $F_2^{-1} = 0$ is the dispersion relation for surface waves of the TE type. It appears, however, that (11) has roots only for real values of α and β if $(\varepsilon\mu - 1)\omega^2 \geq \pi^2/4$. Hence, in the long-wavelength limit, surface waves of the TE type are nonpropagating and therefore do not contribute to the power transport.

From the above analysis it follows that for calculating the power carried by surface waves and space waves, $\boldsymbol{j}''(\alpha,\beta)$ need only be known for small values of α and β. In space coordinates this means that we need only know the average currents, flowing in the x and y directions, rather than the complete current distribution.

Finally, if the microstrip is infinitely long, part of the furnished power may be propagated by the microstrip. We shall discuss this situation in the next section, where it is shown that the contribution to W is in that case due to singularities in $\boldsymbol{j}''(\alpha,\beta)$.

Radiation of Power from an Infinitely Long Transmission Line Excited by a Voltage Slit

We consider a microstrip of infinite length, extending from $x = -\infty$ to $x = \infty$ and from $y = -w/2$ to $y = w/2$ (see Fig. 1). At $x = 0$ a voltage step $U(x)$ is applied. Here U is the unit-step function. We introduce the transmission line concepts of specific inductance L, specific capacitance C, characteristic impedance $Z_c = (L/C)^{1/2}$, and propagation velocity $v = (LC)^{-1/2}$. The permittivity and permeability of the dielectric wafer are again denoted by ε and μ, respectively. The chosen thickness of the wafer is unity and in the half-space $z > 1$ the chosen permittivity and permeability are unity also. Consequently ω, ε, μ, L, and C are dimensionless parameters. The total current J flowing in the x direction is the well-known transmission line solution

$$J(x) = (1/2Z_c) \exp(-ik|x|) \tag{12}$$

where the wavenumber k is related to ω and v by

$$k = \omega/v.$$

The longitudinal current density $j_1(x,y)$ is related to $J(x)$ by

$$\int_{-w/2}^{w/2} j_1(x,y)\, dy = J(x). \tag{13}$$

From (2), (12), and (13) it follows that the Fourier transform $j_1''(\alpha,\beta)$ of $j_1(x,y)$ satisfies

$$j_1''(\alpha,0) = -k/4\pi^2 Z_c(\alpha^2 - k^2) \tag{14}$$

where, in view of (7),

$$\mathrm{Im}\,(k) \leq -0.$$

As mentioned in the introduction, the contribution of the transverse current component j_2'' to W is negligible. In order to show this we first remark that outside the region of excitation, where $j_1(x,y)$ is approximately equal to the magnetostatic current distribution, the ratio $j_1(x,y)/j_1(x,0)$ is an even, nonnegative function of y. From this it follows that, for small values of α and β, the ratio of the Fourier transforms $j_1''(\alpha,\beta)/j_1''(\alpha,0)$ satisfies the relation

$$j_1''(\alpha,\beta)/j_1''(\alpha,0) = 1 - a^2\beta^2 \tag{15}$$

with

$$a^2 < w^2/8.$$

A similar argument applies to the surface charge density in the conducting strip. Let $\rho''(\alpha,\beta)$ be the Fourier transform of the surface charge density $\rho(x,y)$. Then, analogous to (15),

$$\rho''(\alpha,\beta)/\rho''(\alpha,0) = 1 - b^2\beta^2$$

with

$$b^2 < w^2/8. \tag{16}$$

On the other hand, the continuity equation for the electric charge requires that, for any α and β,

$$-i\alpha j_1'' - i\beta j_2' = -i\omega\rho''. \tag{17}$$

From (15)–(17) we then find that, for small values of α and β,

$$j_2''/j_1'' = \alpha\beta(a^2 - b^2) \tag{18}$$

with

$$a^2 < w^2/8$$
$$b^2 < w^2/8.$$

From (18) we conclude that, in the long-wavelength limit, the contribution from j_2'' to W is indeed negligible. Hence the expression for the power density $I(\alpha,\beta)$, defined in (6), simplifies to

$$I(\alpha,\beta) = -4\pi^2 j_1''^* j_1'' Z_{11}^*. \tag{19}$$

We are now in a position to calculate the various contributions to W.

We observe that, because we consider only small values of γ, it is legitimate to replace $\gamma/\tanh\gamma$ by unity. Z_{11} then becomes a rational function of γ_0, having poles approximately at $\gamma_0 = (\varepsilon\mu - 1)\omega^2/\varepsilon$, at $\gamma_0 = -\varepsilon$, and at $\gamma_0 = -1/\mu$. The residues at these poles are easily found and Z_{11} can be decomposed into partial fractions:

$$Z_{11} \approx -\frac{\alpha^2}{i\omega\varepsilon}\left\{\frac{\varepsilon}{\varepsilon+\gamma_0} - \frac{(\varepsilon\mu-1)^2\omega^2/\varepsilon}{\gamma_0 - (\varepsilon\mu-1)\omega^2/\varepsilon}\right\} - \frac{i\omega}{\gamma_0 + 1/\mu},$$
$$|\gamma| \ll 1,\ |\gamma_0| \ll 1. \tag{20}$$

The various contributions of I to W can now be calculated analytically. Consider the region $\alpha^2 + \beta^2 \leq \omega^2$. In this region γ_0 is positive imaginary. As pointed out in the previous section, the contribution to W, which we denote by W_1, can be interpreted as the power radiated into the half-space $z > 1$. We write the result of our calculation in the following form:

$$W_1 = \frac{\omega^2\mu^2}{8\pi L^2}\{f_1(LC,\varepsilon\mu) - f_2(LC,\varepsilon,\mu,\omega)\} \tag{21}$$

with

$$f_1(a,b) = (a^2/b^2)\Bigg[1 + (b^2 - 4b + 1 + 2a) \cdot \left\{\frac{1}{2(a-1)} + \frac{1}{4(a)^{1/2}}\ln\frac{a^{1/2}-1}{a^{1/2}+1}\right\} + \frac{(b^2-a^2)}{a}\left\{\frac{1}{2(a-1)} - \frac{1}{4(a)^{1/2}}\ln\frac{a^{1/2}-1}{a^{1/2}+1}\right\}\Bigg]$$

$$f_2(LC,\varepsilon,\mu,\omega) = \frac{\pi(\varepsilon\mu-1)^3}{2\varepsilon^3\mu^2}\left(\frac{LC}{LC-1}\right)^{3/2}|\omega|.$$

In (21) terms containing ω to the fourth power and higher are omitted because these terms are of the same order as the error introduced by the long-wavelength approximation. The factor f_1 is of the order of unity for all possible values of LC and $\varepsilon\mu$. In order to illustrate this we give some typical values:

$$f_1(1+0,\, 1+0) = 1$$
$$f_1(\infty,\, \infty) = 4/3$$
$$f_1(6,9) = 1.369.$$

The factor f_2 approaches zero if ω goes to zero.

We note that in the dimensionless expression (21) dimensions can be restored by making the following substitutions:

$$W_1 \to W_1(\mu_0/\varepsilon_0)^{1/2}$$
$$\omega \to \omega(\varepsilon_0\mu_0)^{1/2}h$$
$$\varepsilon \to \varepsilon/\varepsilon_0$$
$$\mu \to \mu/\mu_0$$
$$L \to L/\mu_0$$
$$C \to C/\varepsilon_0$$

where ε_0 and μ_0 are the permittivity and permeability, respectively, of the half-space $z > 1$ and h is the wafer thickness.

Next we consider the region $\alpha^2 + \beta^2 > \omega^2$. In that region γ_0 is real and, hence, contributions to W can arise only from the poles of Z_{11} and of $j_1''^* j_1''$. As pointed out in the previous section the pole of Z_{11} at $\gamma_0 \approx (\varepsilon\mu - 1)\omega^2/\varepsilon$ gives a contribution that can be interpreted as the power carried by surface waves of the TM type. We denote this contribution by W_2. For W_2 we find

$$W_2 = \frac{\omega^2\mu^2}{4\pi L^2} f_2(LC,\varepsilon,\mu,\omega) \qquad (22)$$

where f_2 is again given by (21). We conclude that the fraction of the radiated power carried by surface waves is of a higher degree in ω than the fraction carried by the plane waves launched in the half-space $z > 1$. Hence, in the long-wavelength limit this fraction is negligible.

Finally we investigate the contribution to W caused by the singularities of $j_1''^* j_1''$ at $\alpha = \pm k$. We note that homogeneous solutions in which the x component of the wave vector is $\pm k$ are the transmission line solutions. Hence the contributions of these singularities to W are equal to the power carried to infinity by the transmission line. The region in the α–β plane that contributes to W is in this case not limited to small values of β, and hence this contribution cannot be found from (6).

However, because the total supplied power W must be

$$W = 1/4Z_c$$

the power propagated by the transmission line is $1/4Z_c - W_1 - W_2$. We remark that the current $J(x)$, given by (2), may be considered to be a superposition of two waves $J_1(x)$ and $J_2(x)$, with

$$J_1(x) = (1/2Z_c)\exp(-ikx), \qquad -\infty < x < \infty$$
$$J_2(x) = (1/2Z_c)\{\exp(ikx) - \exp(-ikx)\}, \qquad x < 0$$
$$J_2(x) = 0, \qquad x > 0.$$

Now the contribution of J_1 to W_1 and W_2 is zero. This is so because the Fourier transform J_1 contains the Dirac δ function: $\delta(\alpha - k)$ and $\mathrm{Re}\,(Z_{11}) = 0$ for $\alpha = k$. On the other hand, J_2 can be interpreted as a traveling wave, incident from $x = -\infty$ and reflected at an open end at $x = 0$. From this remark it follows that $W_1 + W_2$ may alternatively be interpreted as the power radiation caused by an open-end reflection of a current wave of amplitude $1/2Z_c$. This observation allows us to compare our results with that obtained by Lewin [2].

We account for an amplitude factor $1/2Z_c$, a factor of one half for the ratio average value/peak value and a factor of 120π for the ratio $(\mu_0/\varepsilon_0)^{1/2}$. In order to avoid confusion with our symbol ε we denote the "effective" dielectric constant used by Lewin by $\varepsilon_{\mathrm{eff}}$. If $\mu = 1$ it is identical to LC in our notation. For the ratio of the powers calculated with the two methods we find

$$\frac{P_{\mathrm{Lewin}}}{120\pi \cdot 8Z_c^2 W_1} = \frac{\varepsilon_{\mathrm{eff}} F_L(\varepsilon_{\mathrm{eff}})}{2f_1(LC,\varepsilon\mu)} \qquad (23)$$

where, according to formula (14) of [1], F_L is given by

$$F_L(\varepsilon_{\mathrm{eff}}) = \frac{\varepsilon_{\mathrm{eff}}}{\varepsilon_{\mathrm{eff}} + 1} - \frac{(\varepsilon_{\mathrm{eff}} - 1)^2}{2\varepsilon_{\mathrm{eff}}\varepsilon_{\mathrm{eff}}^{1/2}} \log \frac{\varepsilon_{\mathrm{eff}}^{1/2} + 1}{\varepsilon_{\mathrm{eff}}^{1/2} - 1}$$

and f_1 is given by (21).

It is interesting to note that in the two special cases $\varepsilon\mu = 1$, $LC = \varepsilon_{\mathrm{eff}} = 1$ and $\varepsilon\mu \to \infty$, $LC = \varepsilon_{\mathrm{eff}} \to \infty$ the ratio (23) is exactly unity. For the case $\varepsilon = 2.8$, $\varepsilon_{\mathrm{eff}} = 2.25$, which is considered by Lewin, we find, from (23),

$$\frac{P_{\mathrm{Lewin}}}{120\pi \cdot 8Z_c^2 W_1} = 1.13.$$

Power Radiation from a Half-Wave Open-End Microstrip Resonator

We consider a microstrip extending from $x = -a$ to $x = a$. Let the longitudinal current $J(x)$ along the strip be given by

$$J(x) = \cos(\pi x/2a), \qquad |x| \le a$$
$$J(x) = 0, \qquad \text{elsewhere.} \qquad (24)$$

The Fourier transform $j_1''(\alpha,\beta)$, defined in (1), is now found to be

$$j_1''(\alpha,\beta) = -\frac{\cos(\alpha a)}{4\pi a(\alpha^2 - \pi^2/4a^2)}, \qquad |\beta w| \ll 1. \qquad (25)$$

The calculation of the power radiated by this half-wave resonator goes along the same lines as that in the preceding section. We substitute (20) and (25) into (6). Integration with respect to β then yields

$$W = W_1 + W_2 \qquad (26)$$

where

$$W_1 = \frac{\omega^2 LC}{2\pi\varepsilon^2}\int_0^1 \frac{\cos^2\{\pi z/2(LC)^{1/2}\}}{(LC - z^2)^2} \cdot \left\{(\varepsilon^2\mu^2 - z^2)(1 - z^2) + 2(\varepsilon\mu - 1)^2 z^2 - \frac{(\varepsilon\mu - 1)^3|\omega|}{\varepsilon(1 - z^2)^{1/2}}\right\} dz$$

$$W_2 = \frac{\omega^2 LC}{\pi\varepsilon^2}\int_0^1 \frac{\cos^2\{\pi z/2(LC)^{1/2}\}}{(LC - z^2)^2}\,\frac{(\varepsilon\mu - 1)^3|\omega|}{\varepsilon(1 - z^2)^{1/2}}\,dz.$$

Here W_1 is again the power radiated into the half-space $z > 1$ and W_2 is the power carried by surface waves. The integrals occurring in (26) can, for instance, be evaluated with a stored-program pocket calculator. An approximate expression, which is in most cases sufficiently accurate, is obtained by neglecting the term containing ω to the third power and replacing the factor

$$\frac{\cos^2\{\pi z/2(LC)^{1/2}\}}{(LC-z^2)^2}$$

by $1/(LC)^2$. We then obtain

$$W \approx \frac{\omega^2}{2\pi\varepsilon^2 LC}\left\{\frac{(2\varepsilon\mu-1)^2}{3}+1/5\right\}. \tag{27}$$

In the long-wavelength limit the error in (27) is smaller than 7 percent.

A useful circuit parameter is the quality factor Q of the resonator, which is defined in the following way. Let E_s be the electromagnetic energy stored in the resonator. In our case E_s is conveniently found from

$$E_s = \tfrac{1}{2}\int_{-a}^{a} J^*JL\,dx. \tag{28}$$

The quality factor Q is then defined by

$$Q = \frac{\omega E_s}{W}. \tag{29}$$

In view of (24), (28), and (29) we then find

$$Q = \frac{\pi(L/C)^{1/2}}{4W} \tag{30}$$

where W is given by (26) or, approximately, by (27).

Power Radiation from a Circular Resonator

We consider a circular resonator of width w and radius a, see Fig. 2. Consistent with the long-wavelength condition we assume that a is large compared to w. Let the total current J along the strip, in terms of the polar coordinates r and ϕ, be given by

$$J(r,\phi) = \exp(-i\phi). \tag{31}$$

Contrary to the current density in the preceding section, $\boldsymbol{j}$ is now a two-dimensional vector. In the long-wavelength limit the radiated power is again independent of the longitudinal current distribution and the transverse currents may be neglected. Hence we write for the components j_1 and j_2 of $\boldsymbol{j}$

$$j_1(r,\phi) = -\exp(-i\phi)\sin(\phi)\delta(r-a)$$
$$j_2(r,\phi) = \exp(-i\phi)\cos(\phi)\delta(r-a). \tag{32}$$

The Fourier transform $\boldsymbol{j}''$ is, in view of (2),

$$j_1''(k,\psi) = (ia/4\pi)\{J_0(ka) + \exp(-2i\psi)J_2(ka)\}$$
$$j_2''(k,\psi) = (a/4\pi)\{J_0(ka) - \exp(-2i\psi)J_2(ka)\} \tag{33}$$

where J_0 and J_2 are the zero order and second order Bessel functions, respectively, of the first kind and k and ψ are related to α and β by $\alpha = k\cos\psi$, $\beta = k\sin\psi$. The impedance matrix Z is now a symmetric square matrix of order two. Its elements Z_{11}, Z_{12}, Z_{21}, and Z_{22} are given by (5).

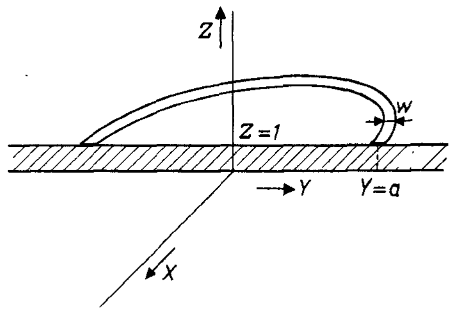

Fig. 2. Cross-sectional view of a circular microstrip resonator.

The evaluation of the average radiated power W is now straightforward. We find

$$W = W_1 + W_2 \tag{34}$$

with

$$W_1 = (\pi\omega^2/\varepsilon^2)\int_0^1 J_1^2(\zeta)\{-z+(\varepsilon\mu-1)^2/z\}$$
$$+\{J_0(\zeta)-J_2(\zeta)\}^2(\varepsilon^2\mu^2 z/4LC)z\,dz$$
$$-(\pi^2|\omega|^3/2\varepsilon^3)(\varepsilon\mu-1)^3J_1^2\{(LC)^{-1/2}\}$$
$$W_2 = (\pi^2|\omega|^3/\varepsilon^3)(\varepsilon\mu-1)^3J_1^2\{(LC)^{-1/2}\}$$
$$\zeta = \{(1-z^2)/LC\}^{1/2}.$$

W_1 is again the fraction of the power carried by space waves and W_2 the fraction carried by surface waves. An approximate expression for W is obtained by replacing $J_1^2(\zeta)$ by $\zeta^2/4$ and $J_0 - J_2$ by unity and omitting the term containing $|\omega|^3$. We then find

$$W \approx (\pi\omega^2/4LC\varepsilon^2)(\varepsilon^2\mu^2 - 4\varepsilon\mu/3 + 8/15). \tag{35}$$

The quality factor Q can now be calculated in a way analogous to that in the preceding section. By using (23) we find

$$Q \approx \frac{4LC(L/C)^{1/2}}{\omega^2\mu^2(1-4/3\varepsilon\mu+8/15\varepsilon^2\mu^2)}. \tag{36}$$

When comparing (27) and (30) with (36) it is seen that the quality factor of a circular resonator is approximately equal to that of a stretched open-end resonator. This is in agreement with results, reported in [7].

Power Radiation from a Hairpin Resonator

The last configuration we investigate is the "hairpin" resonator, drawn in Fig. 3. We assume that $s \ll a$, so that we may represent the current density $\boldsymbol{j}(x,y)$ by

$$j_1(x,y) = \cos(\pi x/2a)\{\delta(y)-\delta(y-s)\}, \qquad 0 \le -x \le a$$
$$j_1(x,y) = 0, \qquad \text{elsewhere}$$
$$j_2(x,y) = \delta(x), \qquad 0 \le y \le s$$
$$j_2(x,y) = 0, \qquad \text{elsewhere}. \tag{37}$$

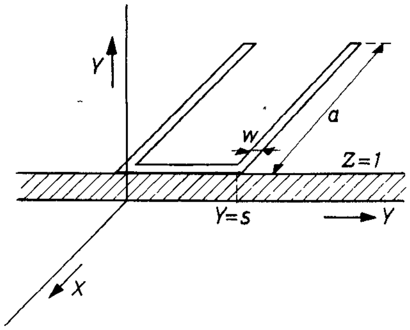

Fig. 3. Cross-sectional view of the hairpin microstrip resonator.

The Fourier transforms $j_1''(\alpha,\beta)$ and $j_2''(\alpha,\beta)$ are now found to be

$$j_1''(\alpha,\beta) = -(i\beta sa/2\pi^2)\frac{2i\alpha a + \pi \exp(-i\alpha a)}{\pi^2 - 4\alpha^2 a^2}$$

$$j_2''(\alpha,\beta) = s/4\pi^2, \qquad \beta s \ll 1. \tag{38}$$

The calculation of the supplied power W is now straightforward. We find

$$W = (\omega^4 s^2/\varepsilon^2)\int_0^1 (A + B + D)\,dz \tag{39}$$

where

$$A = \frac{z^2 - 2z(LC)^{1/2}\sin\{\pi z/2(LC)^{1/2}\} + LC}{32\pi(LC - z^2)^2} \cdot \{(1 - z^2)^2(\varepsilon\mu - z^2) + 4(1 - z^2)(\varepsilon\mu - 1)^2 z^2 + (8/\varepsilon)(\varepsilon\mu - 1)^3|\omega|(1 - z^2)^{1/2}z^2\}$$

$$B = \frac{z(LC)^{1/2}\sin\{\pi z/2(LC)^{1/2}\} - z^2}{16\pi(LC - z^2)} \cdot \{-(1 - z^2)^2 + 4(1 - z^2)(\varepsilon\mu - 1)^2 + (8/\varepsilon)(\varepsilon\mu - 1)^3(1 - z^2)^{1/2}|\omega|\}$$

$$D = (1/4\pi)\{(\varepsilon\mu - 1)^2(1 - z^2) + (\varepsilon\mu - 1)^3(1 - z^2)^{1/2}|\omega|/\varepsilon + (1 - z^2)(-5 + z^2 + 8\varepsilon\mu)/8\}.$$

We note that L and C are now the specific inductance and specific capacitance, respectively, for the mode of propagation with odd symmetry. It appears that the term $\int_0^1 D\,dz$ is several times larger than the term $\int_0^1 (A + B)\,dz$. Using $W \approx (\omega^4 s^2/\varepsilon^2)\int_0^1 D\,dz$ we obtain the approximate relation

$$W \approx (\omega^4 s^2/4\pi\varepsilon^2)\{4/15 + (2/3)(\varepsilon\mu - 1)\varepsilon\mu + (\pi/4\varepsilon)(\varepsilon\mu - 1)^3|\omega|\}. \tag{40}$$

The quality factor is again given by (30) and hence, in view of (40), we obtain

$$Q \approx (\pi^2\varepsilon^2/\omega^4 s^2)(L/C)^{1/2} \cdot \{4/15 + (2/3)(\varepsilon\mu - 1)\varepsilon\mu + (\pi/4\varepsilon)(\varepsilon\mu - 1)^3\}^{-1}. \tag{41}$$

Contrary to the preceding cases, where, in the long-wavelength limit, Q was proportional to ω^{-2}, in this example Q is proportional to ω^{-4}. Hence, in the long-wavelength limit, the hairpin resonator is expected to exhibit particularly low radiation losses. The gain in Q may, however, be smaller than expected from (41), due to ohmic losses.

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